

# POLYLOGARITHMS, HYPERFUNCTIONS AND GENERALIZED LIPSCHITZ SUMMATION FORMULAE

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ABSTRACT. A generalization of the classical Lipschitz summation formula is proposed. It involves new polylogarithmic rational functions constructed via the Fourier expansion of certain sequences of Bernoulli–type polynomials. Related families of one–dimensional hyperfunctions are also constructed.

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## 1. INTRODUCTION

The purpose of this paper is to provide a natural setting which allows to generalize the Lipschitz summation formula to negative powers.

We recall that the classical Lipschitz summation formula gives the Fourier series expansion of the periodic analytic function obtained by summation over integer translates of the power  $z^{-k}$ , where  $z \in \mathfrak{H}$  (the complex upper half plane) and  $k$  is a positive integer.

By introducing suitable generalizations of the Lipschitz formula, we will show how to construct new classes of hyperfunctions. Indeed, each of these generalized formulae will provide a hyperfunctional equation involving a specific two–variable polylogarithm series.

A simple introduction to the theory of hyperfunctions is sketched in Section 2, where the interested reader is referred to the classical works [15], [7]. Hyperfunctional cohomology groups have also been related to automorphic forms and period functions [3].

The main results of this paper are essentially two. The first result is a generalization of the classical Lipschitz summation formula. The second one is the connection we establish between a new class of polylogarithms, the theory of hyperfunctions in one variable and that of polynomial structures of Appell type.

Precisely, our construction can be summarized as follows. We will define suitable sequences of Appell polynomials of Bernoulli type: indeed, they share with the Bernoulli polynomials several arithmetic properties, including certain famous congruences (see the Appendix). Their periodic versions provide primitives of the periodic delta function. To each sequence we will associate a natural extension of the notion of polylogarithmic function defined in the unit disc. We call this extension a *delta rational function*. It is a two-variable Dirichlet series, extending to the whole Riemann sphere as a meromorphic function. The case of the classical Bernoulli polynomials corresponds to the standard polylogarithmic function, and it is treated thoroughly. The new Appell polynomial sequences we construct,  $\{P_n(x)\}_{n \in \mathbb{N}}$  and  $\{Q_n(x)\}_{n \in \mathbb{N}}$  (depending on certain parity properties) and the generalized polylogarithms  $\delta_n(q)$  proposed here satisfy interesting hyperfunctional equations. These equations in turn correspond to generalized Lipschitz summation formulae. For instance, in the case of polynomials  $\{P_n(x)\}_{n \in \mathbb{N}}$  it reads

$$(1) \quad \sum_{k \in \mathbb{Z}} \varphi_{\overline{P}_n}(\tau + k) = 2i(2\pi i)^{-n} \begin{cases} \Delta_{-n}(q) & \text{if } |q| < 1, \text{ i.e. } \Im \tau > 0 \\ (-1)^{n-1} \Delta_{-n}(q^{-1}) & \text{if } |q| > 1, \text{ i.e. } \Im \tau < 0 \end{cases},$$

where  $\overline{P}_n$  are the hyperfunctions associated to the polynomial  $P_n$  and  $\varphi_{\overline{P}_n}$  is the function in  $O^1(\overline{\mathbb{C}} \setminus [0, 1])$  representing  $\overline{P}_n$ .

In the last part of the paper, the relation between the theory of formal groups and that of hyperfunctions is clarified. We will show that the Appell structures introduced in the previous construction can be viewed as polynomial realizations of certain formal group laws, related with the universal Lazard formal group.

The future research plans include an extension of the proposed construction to the multidimensional case (see also [12] for a different generalization), as well as possible applications of the proposed Lipschitz formulae to the study of Eisenstein series and periods of modular forms.

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## 2. HYPERFUNCTIONS: BASIC PRELIMINARIES

In this section, we will provide a brief and self-consistent introduction to the theory of hyperfunctions of a single variable, following closely [15], Chapter IX of [7], [10] and [9]. For a more extensive treatment and further details, the reader is invited to consult these books, as well as [13] for interesting applications.

Let us denote by  $\mathcal{O}$  the sheaf of holomorphic functions on  $\mathbb{C}$ , and denote by  $\mathbb{C}^+$  and  $\mathbb{C}^-$  the upper and lower half planes of  $\mathbb{C}$ .

**Definition 1.** *The space of hyperfunctions  $\mathcal{B}$  on the real line  $\mathbb{R}$  is*

$$(2) \quad \mathcal{B}(\mathbb{R}) := H_{\mathbb{R}}^1(\mathbb{C}, \mathcal{O}),$$

*i.e. is the first sheaf cohomology group on  $\mathbb{R}$ .*

Now, since  $\mathbb{C}^+ \cup \mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}$ , we have the following decomposition:

$$(3) \quad H_{\mathbb{R}}^1(\mathbb{C}, \mathcal{O}) = [H^0(\mathbb{C}^+, \mathcal{O}) \oplus H^0(\mathbb{C}^-, \mathcal{O})] / H^0(\mathbb{C}, \mathcal{O}).$$

In other words, since  $H^0(\mathcal{O})$  represents nothing but the global sections of the sheaf (i.e. holomorphic functions), a hyperfunction can be thought as a pair of holomorphic functions on the upper and lower half planes respectively, modulo an entire function. This geometric definition immediately generalizes to the case of hyperfunctions in several variables. If  $\Omega \subset \mathbb{R}$  is an open set, and  $U$  an arbitrary complex neighborhood of  $\Omega$ , then clearly

$$(4) \quad \mathcal{B}(\Omega) = H_{\Omega}^1(U, \mathcal{O}).$$

Still denoting by  $\mathcal{O}(U)$  the space of holomorphic functions in  $U$ , another equivalent definition is

$$(5) \quad \mathcal{B}(\Omega) := \varinjlim_{U \supset \Omega} \mathcal{O}(U \setminus \Omega) / \mathcal{O}(\Omega)$$

where the inductive limit with respect to the family of complex neighborhoods  $U \supset \Omega$  is considered. A hyperfunction  $f(x)$  is therefore an equivalence class  $[F(z)]$ , whose representative is  $F(z) \in \mathcal{O}(U \setminus \Omega)$ . The representative  $F(z)$  is said to be a defining function of  $f(x)$ . Since

$$U \supset \Omega = U_+ \cup U_-, \quad U_{\pm} = U \cap \{\Im z \leq 0\},$$

often the following boundary-value representation is used:

$$f(x) = F_+(x + i0) - F_-(x - i0),$$

with  $F_{\pm}(z) = F(z)|_{U_{\pm}}$ .

Alternatively, one can construct a theory of hyperfunctions based on analytic functionals. Let  $K \subset \mathbb{C}$  be a non empty compact set, and denote by  $A$  the space of entire analytic functions in  $\mathbb{C}$ .

**Definition 2.** *The space  $A'(K)$  of the analytic functionals carried by  $K$  is the space of linear forms  $u$  acting on  $A$  such that for every neighborhood  $V$*

of  $K$  there is a constant  $C_V > 0$  such that

$$(6) \quad |u(\varphi)| \leq C_V \sup_V |\varphi|, \quad \forall \varphi \in A.$$

Observe that  $A'(K)$  is a Fréchet space, since a seminorm is associated to each neighborhood  $V$  of  $K$ . One can define

$$(7) \quad \mathcal{B}(\Omega) := A'(\overline{\Omega}) / A'(\partial\Omega).$$

It is interesting to notice that the space of hyperfunctions  $\psi \in B(\Omega)$  with compact support  $K \subset \Omega$  can be identified with analytic functionals in  $A'(\mathbb{R})$  with support  $K$ . Indeed, an analytic functional  $u$  on  $\cup_{i=1}^r K_i$  can always be decomposed into a sum  $u = u_1 + \dots + u_r$ , with each of the functionals  $u_j \in A'(K_j)$ . Consequently, since  $\text{supp } \psi \subset K \cup \partial\Omega$ , the contribution of  $\psi$  on  $\partial\Omega$  can be factored out and  $\psi$  is identified with an uniquely defined functional with support in  $\Omega$ .

Therefore, we can also think of  $A'(K)$  as the space of hyperfunctions with support in  $K$ . The link between the two approaches to the theory of hyperfunctions is now provided by the following lemma. Let  $\mathcal{O}^1(\overline{\mathbb{C}} \setminus K)$  denotes the space of holomorphic functions on  $\overline{\mathbb{C}} \setminus K$  and vanishing at infinity.

**Lemma 3.** *The spaces  $A'(K)$  and  $\mathcal{O}^1(\overline{\mathbb{C}} \setminus K)$  are canonically isomorphic. To each  $u \in A'(K)$  it corresponds a function  $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus K)$  given by*

$$\varphi(z) = u(c_z), \quad \forall z \in \mathbb{C} \setminus K,$$

where  $c_z(x) = \frac{1}{\pi} \frac{1}{x-z}$ . Conversely, to each  $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus K)$  it corresponds the hyperfunction

$$(8) \quad u(\psi) = \frac{i}{2\pi} \int_{\gamma} \varphi(z) \psi(z) dz, \quad \forall \psi \in A,$$

where  $\gamma$  is any piecewise  $\mathcal{C}^1$  path winding around  $K$  in the positive direction.

For future purposes, we also briefly describe *periodic hyperfunctions*. Let  $\mathbb{T}^1 = \mathbb{R} \setminus \mathbb{Z} \subset \mathbb{C} \setminus \mathbb{Z}$ . A hyperfunction on  $\mathbb{T}^1$  is a linear functional  $\Psi$  on the space  $\mathcal{O}(\mathbb{T}^1)$  of functions analytic in a complex neighborhood of  $\mathbb{T}^1$  such that for all neighborhood  $V$  of  $\mathbb{T}^1$  there exists a constant  $C_V > 0$  such that

$$(9) \quad |\Psi(\Phi)| \leq C_V \sup_V |\Phi|, \quad \forall \Phi \in \mathcal{O}(V).$$

We will denote by  $A'(\mathbb{T}^1)$  the Fréchet space of hyperfunctions with support in  $\mathbb{T}^1$ . Let  $\mathcal{O}_{\Sigma}$  denote the complex vector space of holomorphic functions  $\Phi : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ , 1-periodic, bounded at  $\pm i\infty$  and such that  $\Phi(\pm i\infty) := \lim_{\Im z \rightarrow \pm\infty} \Phi(z)$  exist and verify  $\Phi(+i\infty) = -\Phi(-i\infty)$ .

**Lemma 4.** *The spaces  $A'(\mathbb{T}^1)$  and  $\mathcal{O}_{\Sigma}$  are canonically isomorphic: to each  $\Psi \in A'(\mathbb{T}^1)$  it corresponds a function  $\Phi \in \mathcal{O}_{\Sigma}$  given by*

$$\Phi(z) = \Psi(C_z), \quad \forall z \in \mathbb{C} \setminus K,$$

where  $C_z(x) = \cot \pi(x - z)$ . Conversely, to each  $\Phi \in \mathcal{O}_\Sigma$  it corresponds the hyperfunction

$$(10) \quad \Psi(\Xi) = \frac{i}{2} \int_{\Gamma} \Phi(z) \Xi(z) dz, \quad \forall \Xi \in A'(\mathbb{T}^1),$$

where  $\Gamma$  is any piecewise  $C^1$  path winding around a closed interval  $I \subset \mathbb{R}$  of length 1 in the positive direction.

Given a compactly supported hyperfunction, making infinitely many copies of it translating its support leads to a periodic hyperfunction. This point of view is systematically exploited in [9], from which the following commutative diagram is taken:

$$(11) \quad \begin{array}{ccc} A'([0, 1]) & \longrightarrow & O^1(\overline{\mathbb{C}} \setminus [0, 1]) \\ \Sigma_{\mathbb{Z}} \downarrow & & \downarrow \Sigma_{\mathbb{Z}} \\ A'(\mathbb{T}^1) & \longrightarrow & \mathcal{O}_\Sigma \end{array}$$

The horizontal lines are the above mentioned isomorphisms, and  $\Sigma_{\mathbb{Z}}$  denotes the summation over integer translates.

### 3. DELTA RATIONAL FUNCTIONS

**3.1. Delta rational functions and polylogarithms.** In this section we introduce the notion of delta rational functions, which represent a natural generalization of the notion of polylogarithms [11], [19]. These functions are connected to the classical Bernoulli polynomials via the Fourier expansion formula (14) and enter a generalized Lipschitz summation formula providing new classes of hyperfunctions.

**Definition 5.** *The delta rational function is the series expressed for any  $n \in \mathbb{Z}$ , and  $q \in \mathbb{D}$  by*

$$(12) \quad \delta_n(q) = \sum_{k=1}^{\infty} k^n q^k.$$

Observe that, if  $n \geq 0$ ,  $\delta_n$  extends to the whole Riemann sphere as a rational function of degree  $n + 1$  with just a simple pole of order  $n + 1$  at  $q = 1$ . If  $n \leq -1$  then  $\delta_n$  coincides with the classical polylogarithm series of order  $-n$  (actually  $\delta_{-1}(q) = -\log(1 - q)$ ) and extends to the whole  $\mathbb{C} \setminus [1, +\infty)$  and as a multiplicative function to the whole  $\mathbb{C} \setminus \{0, 1, \infty\}$ . Indeed, since

$$q \partial_q \delta_n(q) = \delta_{n+1}(q),$$

one can define the analytic continuation of  $\delta_n$ . For instance, the continuation of  $\delta_{-2}$  is obtained by means of the integral formula

$$\delta_{-2}(q) = - \int_0^q \frac{\log(1-t)}{t} dt = \int_0^q \left( \int_0^t \frac{d\zeta}{1-\zeta} \right) \frac{dt}{t}.$$

Note that  $[1, +\infty)$  is a branch cut. For all  $n \in \mathbb{Z}$  and  $q \in \mathbb{D}$ , one has

$$(13) \quad \delta_n(q^k) = k^{-1-n} \sum_{\Lambda^k=1} \delta_n(\Lambda q),$$

where  $\Lambda$  denotes a  $k$ -th root of unity. The equality extends to the closed disk if  $n \leq -2$ . One can directly prove the following result.

**Lemma 6.** *The fundamental inversion equation holds*

$$(14) \quad \delta_n(q) + (-1)^n \delta_n(q^{-1}) = \begin{cases} 0 & \text{if } n \geq 1 \\ -\frac{(2\pi i)^{-n}}{(-n)!} B_{-n}\left(\frac{\log q}{2\pi i}\right) & \text{if } n \leq 0 \end{cases},$$

for all  $q \neq 1$  if  $n \geq 0$ , and  $q$  in  $\mathbb{C} \setminus [0, +\infty]$  if  $n \leq 0$ . Here  $B_k$  is the  $k$ -th Bernoulli polynomial.

**3.2. Periodic hyperfunctions.** The inversion relation (14) has a beautiful interpretation in terms of hyperfunctions, as we will show below.

**Definition 7.** *The periodic Bernoulli functions and distributions are expressed by*

$$(15) \quad \tilde{B}_n(x) = \begin{cases} -\frac{B_n(x-[x])}{n!} & \text{if } n \geq 1 \\ -1 + \delta_{\mathbb{T}}(x) & \text{if } n = 0 \\ \delta_{\mathbb{T}}^{(-n)}(x) & \text{if } n < 0 \end{cases},$$

where  $[x]$  is the integer part of  $x$ ,  $\delta_{\mathbb{T}}$  is the periodic delta distribution and  $\delta_{\mathbb{T}}^{(k)}$  its derivative of order  $k \geq 0$ .

Explicitly,

$$(16) \quad \delta_{\mathbb{T}}(x) = \sum_{k=-\infty}^{+\infty} e^{2\pi i k x}.$$

From the well-known property of the periodic Bernoulli functions

$$(17) \quad \frac{d^n}{dx^n} (B_n(x-[x])) = n! (1 - \delta_{\mathbb{T}}),$$

we see that we can consider the Bernoulli functions as primitives of the periodic delta function. It is immediate to check that the following hyperfunctional equation holds.

**Proposition 8.** *Let  $q^{\pm} = e^{2\pi i(x \pm i0)}$ ,  $x \in \mathbb{R}$ . For all  $n \in \mathbb{Z}$  we have*

$$(18) \quad \delta_n(q^+) + (-1)^n \delta_n((q^-)^{-1}) = (2\pi i)^{-n} \tilde{B}_{-n}(x).$$

Note that on both sides one can apply derivatives  $q\partial_q$  and  $(2\pi i)^{-1}\partial_x$ .

**3.3. Generalized Lipschitz summation formula.** As is well known, the classical Lipschitz formula is a consequence of the Poisson summation formula, relating sums over integers of pairs of Fourier transforms. It states that

$$(19) \quad \sum_{n \in \mathbb{Z}} \frac{1}{(n+z)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i r z}, \quad z \in \mathbb{H}, \quad k \in \mathbb{Z}_{\geq 2}$$

(for a proof of (19), see, for instance, D. Zagier, Chapter 4 in [18]).

In this section, we will give an hyperfunctional generalization of (19). We make use of the fact that periodic hyperfunctions with compact support are the result of a summation over integer translates of hyperfunctions with support on  $[0, 1]$ , according to the commutative diagram (11). By applying the results described in the previous section, we get a functional version of the Lipschitz formula.

**Theorem 9.** *For all  $n \in \mathbb{Z}$  we have*

$$(20) \quad \sum_{k \in \mathbb{Z}} \varphi_{\overline{B}_n}(\tau + k) = 2i(2\pi i)^{-n} \begin{cases} \delta_{-n}(q) & \text{if } |q| < 1, \text{ i.e. } \Im \tau > 0 \\ (-1)^{n-1} \delta_{-n}(q^{-1}) & \text{if } |q| > 1, \text{ i.e. } \Im \tau < 0 \end{cases},$$

whereas the usual Lipschitz formula corresponds to  $n \leq -1$ . Here  $\overline{B}_n$  is the restriction to  $[0, 1]$  of  $\widetilde{B}_n$  and  $\varphi_{\overline{B}_n}$  is the function in  $O^1(\overline{\mathbb{C}} \setminus [0, 1])$  which represents the hyperfunction  $\overline{B}_n$

$$(21) \quad \varphi_{\overline{B}_n}(\tau) = \left\langle \overline{B}_n, \frac{1}{\pi} \frac{1}{x - \tau} \right\rangle_{[0,1]}.$$

*Proof.* It is enough to remark that an inverse of the operator  $\sum_{\mathbb{Z}}$  on the hyperfunctions is just the restriction of a periodic hyperfunction to the interval  $[0, 1]$ . Thus the r.h.s. of (18) reads

$$(22) \quad \sum_{k \in \mathbb{Z}} (2\pi i)^{-n} \overline{B}_{-n}(x + k)$$

and from (14) and (18) one obtains the desired formula by considering the two associated holomorphic functions.  $\square$

Explicitly, formula (21) reads

$$(23) \quad \varphi_{\overline{B}_n}(\tau) = \begin{cases} \left\langle \delta^{(-n)}(x), \frac{1}{\pi} \frac{1}{x - \tau} \right\rangle_{[0,1]} = (-1)^{-n+1} \frac{(-n)!}{\pi \tau^{-n+1}} & \text{if } n < 0 \\ \left\langle -1 + \delta(x), \frac{1}{\pi} \frac{1}{x - \tau} \right\rangle_{[0,1]} = -\frac{1}{\pi \tau} [1 + \tau \log(1 - 1/\tau)] & \text{if } n = 0 \\ \left\langle -\frac{1}{n!} B_n(x), \frac{1}{\pi} \frac{1}{x - \tau} \right\rangle_{[0,1]} = -\frac{1}{\pi n!} [B_n(\tau) \log(1 - 1/\tau) + R_n(\tau)] & \text{if } n > 0 \end{cases},$$

where  $R_{n-1}$  is the polynomial of degree  $n-1$  such that  $B_n(\tau) \log(1-1/\tau) + R_n(\tau) \in O^1(\overline{\mathbb{C}} \setminus [0, 1])$ . Here are the first six polynomials

$$(24) \quad \begin{aligned} R_1(\tau) &= 1, \quad R_2(\tau) = \tau - \frac{1}{2}, \quad R_3(\tau) = \tau^2 - \tau + \frac{1}{12}, \\ R_4(\tau) &= \tau^3 - \frac{3}{2}\tau^2 + \frac{1}{3}\tau + \frac{1}{12}, \quad R_5(\tau) = \tau^4 - 2\tau^3 + \frac{3}{4}\tau^2 + \frac{1}{4}\tau - \frac{13}{360}, \\ R_6(\tau) &= \tau^5 - \frac{5}{2}\tau^4 + \frac{4}{3}\tau^3 + \frac{1}{2}\tau^2 - \frac{13}{60}\tau - \frac{7}{120}. \end{aligned}$$

**Remark.** One could object that we should have used the simplest Appell sequence  $\{x^n\}_{n \in \mathbb{N}}$  and their associated hyperfunctions on  $[0, 1]$  to write the generalized Lipschitz summation formula. But this would trivially give zero, since

$$(25) \quad x^n = \frac{1}{n+1} [B_{n+1}(x+1) - B_{n+1}(x)]$$

and the function  $B_{n+1}(x+1) - B_{n+1}(x)$  clearly belongs to the kernel of  $\sum_{\mathbb{Z}}$ . Instead, other less trivial Appell sequences can be used to provide useful generalizations of the construction we proposed, as will be illustrated in the subsequent sections.

#### 4. THE GENERAL CASE: SEQUENCES OF APPELL POLYNOMIALS AND HYPERFUNCTIONS

**4.1. Appell sequences.** By analogy with the theory developed in the previous section, we study a more general class of polynomials of Bernoulli type, with the aim of constructing associated families of hyperfunctions. The main request is that they belong to the class of Appell polynomials, i.e. polynomials satisfying the property

$$(26) \quad \frac{d}{dx} A_n(x) = n A_{n-1}(x),$$

with the normalization

$$(27) \quad A_0(x) = \text{const.}$$

**Lemma 10.** *The Fourier series expansion of the polynomials (26), for  $0 < x < 1$  and  $n \geq 1$  has the form*

$$(28) \quad A_n(x) = \sum_{k=-\infty}^{\infty} c_k(n) e^{2\pi i k x},$$

with

$$(29) \quad c_k = \int_0^1 A_n(t) e^{-2\pi i k t} dt.$$

We get

$$(30) \quad A_n(x) = -n! \sum_{k=1}^{\infty} \left[ \sum_{j=1}^n \frac{1}{j!} \frac{\varphi_j}{(2\pi ik)^{n+1-j}} e^{2\pi ikx} + \sum_{j=1}^n (-1)^{n+1-j} \frac{1}{j!} \frac{\varphi_j}{(2\pi ik)^{n+1-j}} e^{-2\pi ikx} \right] + c_0,$$

with  $0 < x < 1$ , and  $\varphi_j = A_j(1) - A_j(0)$ ,  $j = 1, \dots, n$ . The standard Bernoulli polynomials correspond to the case  $c_0 = 0$ ,  $\varphi_1 = 1$ ,  $\varphi_j = 0$ ,  $j = 2, 3, \dots$

*Proof.* It is a direct consequence of the conditions (26)–(29) and of the formula of integration by parts.  $\square$

If  $\varphi_j = 0$  for  $j$  even or  $j$  odd, then formula (30) can be written in a more compact form. The corresponding polynomial sequences will be denoted by  $\{p_n(x)\}_{n \in \mathbb{N}}$  and  $\{q_n(x)\}_{n \in \mathbb{N}}$ , respectively.

a) If  $\varphi_j = 0$  for  $j$  even, we define

$$(31) \quad p_n(x) =: -n! \sum_{k=1}^{\infty} \sum_{\substack{j=1 \\ j \text{ odd}}}^n \frac{1}{j!} \frac{\varphi_j}{(2\pi ik)^{n+1-j}} \left[ e^{2\pi ikx} + (-1)^n e^{-2\pi ikx} \right] + c_0, \quad 0 < x < 1.$$

b) If  $\varphi_j = 0$  for  $j$  odd, we introduce

$$(32) \quad q_n(x) =: -n! \sum_{k=1}^{\infty} \sum_{\substack{j=1 \\ j \text{ even}}}^n \frac{1}{j!} \frac{\varphi_j}{(2\pi ik)^{n+1-j}} \left[ e^{2\pi ikx} + (-1)^{n+1} e^{-2\pi ikx} \right] + c_0, \quad 0 < x < 1.$$

**Remark.** The previous conditions on  $\varphi_j$  are not particularly restrictive. Indeed, one can easily construct infinitely many polynomial sequences possessing the prescribed parity properties. Their generating function has the general form

$$(33) \quad \frac{te^{xt}}{(e^t - 1)g(t)} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!},$$

where  $g(t)$  is any real analytic function such that  $\lim_{t \rightarrow 0} g(t) = 1$  and satisfying the parity condition  $g(t) = g(-t)$  for the case a) and  $g(t) = -g(-t)$  for the case b).

### Examples

a) Take

$$(34) \quad \frac{t^2 e^{xt}}{(e^t - 1) \sin t} = \sum_{n=0}^{\infty} a_n(x) \frac{t^n}{n!}$$

We easily deduce

$$\begin{aligned}
a_0(x) &= 1, \quad a_1(x) = x - \frac{1}{2}, \quad a_2(x) = x^2 - x + \frac{1}{2}, \\
a_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{1}{2}, \\
(35) \quad a_4(x) &= x^4 - 2x^3 + 3x^2 - 2x + \frac{23}{30}, \dots
\end{aligned}$$

b) Consider the generating function

$$(36) \quad \frac{te^{xt}}{(e^t - 1) \cos t} = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}$$

We immediately get

$$\begin{aligned}
b_0(x) &= 1, \quad b_1(x) = x - \frac{1}{2}, \quad b_2(x) = x^2 - x + \frac{7}{6}, \\
b_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{7}{2}x - \frac{3}{2}, \\
(37) \quad b_4(x) &= x^4 - 2x^3 + 7x^2 - 6x + \frac{179}{30}, \dots
\end{aligned}$$

**4.2. Extended delta rational functions.** Let  $n \in \mathbb{Z}$ ,  $q \in \mathbb{D}$ . A straightforward generalization of  $\delta_n(q)$  adapted to the chosen Appell polynomials is provided by the extended delta rational function  $\Delta_n(q)$ .

**Definition 11.** *The extended delta rational function, for any  $n \in \mathbb{Z}$  and  $q \in \mathbb{D}$ , is defined by*

$$(38) \quad \Delta_n(q) = \begin{cases} \sum_{k=1}^{\infty} k^n q^k & n > 0 \\ \sum_{k=1}^{\infty} a_k(n) q^k & n \leq 0 \end{cases}$$

where

$$(39) \quad a_k(n) = \sum_{\substack{j \\ j \text{ even or odd}}}^n \frac{1}{j!} \frac{\varphi_j}{k^{n+1-j}}.$$

In eq. (39), the summation should be understood either over the even values of  $j$  or over the odd ones, depending on the choice of the polynomials (31) or (32), respectively. The above definition is motivated by the following result, which provides an extension of the construction proposed in Section 3. Our aim is to obtain the hyperfunctional equations associated to the proposed Appell polynomials. As a consequence of the Fourier expansion (30) and of relations (31)–(32) we get the relation between extended delta rational functions and Appell sequences.

**Lemma 12.** *The following inversion equations, generalizing relation (14), hold*

$$(40) \quad \Delta_n(q) + (-1)^n \Delta_n(q^{-1}) = \begin{cases} 0 & \text{if } n \geq 1 \\ -\frac{(2\pi i)^{-n}}{(-n)!} P_{-n}\left(\frac{\log q}{2\pi i}\right) & \text{if } n \leq 0 \end{cases},$$

and

$$(41) \quad \Delta_n(q) + (-1)^{n+1} \Delta_n(q^{-1}) = \begin{cases} 0 & \text{if } n \geq 1 \\ -\frac{(2\pi i)^{-n}}{(-n)!} Q_{-n}\left(\frac{\log q}{2\pi i}\right) & \text{if } n \leq 0 \end{cases},$$

where  $P_n$  and  $Q_n$  are respectively the Appell polynomials (31) and (32), to which the constant  $c_0$  has been subtracted. The inversion relations hold for all  $q \neq 1$  if  $n \geq 0$ , whereas for  $n \leq 0$   $q$  can be taken in  $\mathbb{C} \setminus [0, +\infty]$ .

By analogy with formulae (15), the associated periodic functions and distributions are defined as

$$(42) \quad \tilde{P}_n(x) = \begin{cases} -\frac{P_n(x-[x])}{n!} & \text{if } n \geq 1 \\ -1 + \delta_{\mathbb{T}}(x) & \text{if } n = 0 \\ \delta_{\mathbb{T}}^{(-n)}(x) & \text{if } n < 0 \end{cases}.$$

with an analogous definition for the case of polynomials  $Q_n(x)$ . For all  $n \in \mathbb{Z}$  we have the hyperfunctional equations

$$(43) \quad \Delta_n(q^+) + (-1)^n \Delta_n((q^-)^{-1}) = (2\pi i)^{-n} \tilde{P}_{-n}(x),$$

and

$$(44) \quad \Delta_n(q^+) + (-1)^{n+1} \Delta_n((q^-)^{-1}) = (2\pi i)^{-n} \tilde{Q}_{-n}(x).$$

The proof of these relations is again a direct consequence of the previous definitions. Also, by denoting with  $\overline{P}_n$  the restriction to  $[0, 1]$  of  $\tilde{P}_n$  and with  $\varphi_{\overline{P}_n}$  the function in  $O^1(\overline{\mathbb{C}} \setminus [0, 1])$  which represents the hyperfunction  $\overline{P}_n$ , namely

$$(45) \quad \varphi_{\overline{P}_n}(\tau) = \left\langle \overline{P}_n, \frac{1}{\pi} \frac{1}{x - \tau} \right\rangle_{[0,1]},$$

we find, for any choice of the sequence  $\{P_n(x)\}_{n \in \mathbb{N}}$  and  $\{Q_n(x)\}_{n \in \mathbb{N}}$  our main result, i.e. the *generalized Lipschitz formula*

$$(46) \quad \sum_{k \in \mathbb{Z}} \varphi_{\overline{P}_n}(\tau + k) = 2i(2\pi i)^{-n} \begin{cases} \Delta_{-n}(q) & \text{if } |q| < 1, \text{ i.e. } \Im \tau > 0 \\ (-1)^{n-1} \Delta_{-n}(q^{-1}) & \text{if } |q| > 1, \text{ i.e. } \Im \tau < 0 \end{cases},$$

and the corresponding formula for  $\overline{Q}_n$  (again the usual Lipschitz formula corresponds to  $n \leq -1$ ). The explicit computation of  $\varphi_{\overline{P}_n}(\tau)$  and  $\varphi_{\overline{Q}_n}(\tau)$  is completely analogous to the proposed construction for the Bernoulli polynomials and we will not repeat it here.

## 5. A CONNECTION WITH THE LAZARD FORMAL GROUP

We describe here briefly an interesting connection between formal groups, hyperfunctions, and the so-called universal Bernoulli polynomials.

Given a commutative ring with identity  $R$ , we will denote by  $R[x_1, x_2, \dots]$  the ring of formal power series in  $x_1, x_2, \dots$  with coefficients in  $R$ . Following [6], [4], we recall that a commutative one-dimensional formal group law over  $R$  is a two-variable formal power series  $\Phi(x, y) \in R[x, y]$  such that

$$1) \quad \Phi(x, 0) = \Phi(0, x) = x$$

and

$$2) \quad \Phi(\Phi(x, y), z) = \Phi(x, \Phi(y, z)).$$

When  $\Phi(x, y) = \Phi(y, x)$ , the formal group is said to be commutative. The existence of an inverse formal series  $\varphi(x) \in R[x]$  such that  $\Phi(x, \varphi(x)) = 0$  follows from the previous definition.

Let us consider the polynomial ring  $\mathbb{Q}[c_1, c_2, \dots]$  and the formal power series

$$(47) \quad F(s) = s + c_1 \frac{s^2}{2} + c_2 \frac{s^3}{3} + \dots$$

Let  $G(t)$  be the associated inverse series

$$(48) \quad G(t) = t - c_1 \frac{t^2}{2} + (3c_1^2 - 2c_2) \frac{t^3}{6} + \dots$$

so that  $F(G(t)) = t$ . The series (47) and (48) are called formal group logarithm and formal group exponential, respectively. The formal group law related to these series is provided by

$$\Phi(s_1, s_2) = G(F(s_1) + F(s_2))$$

and it represents the so-called *Lazard's Universal Formal Group* [6]. It is defined over the Lazard ring  $L$ , i.e. the subring of  $\mathbb{Q}[c_1, c_2, \dots]$  generated by the coefficients of the power series  $G(F(s_1) + F(s_2))$ .

In [16], the *universal Bernoulli polynomials*  $B_n^G(x, c_1, \dots, c_k, \dots) \equiv B_k^G(x)$  related to the formal group exponential  $G$  have been introduced. They are defined by

$$(49) \quad \frac{t}{G(t)} e^{xt} = \sum_{n \geq 0} B_n^G(x) \frac{t^n}{n!}, \quad x \in \mathbb{R}.$$

The corresponding numbers by construction coincide with the universal Bernoulli numbers discovered in [5], and generated by

$$(50) \quad \frac{t}{G(t)} = \sum_{n \geq 0} \widehat{B}_n \frac{t^n}{n!}, \quad x \in \mathbb{R}.$$

Observe that when  $a = 1$ ,  $c_i = (-1)^i$ , then  $F(s) = \log(1+s)$ ,  $G(t) = e^t - 1$ , and the universal Bernoulli polynomials and numbers reduce to the standard ones. Many other examples of Bernoulli-type polynomials considered in the literature are obtaining by specializing the rational coefficients  $c_i$ . The reasons to consider such a generalization of Bernoulli polynomials are manifold. First, to any choice of the coefficients  $c_1, c_2, \dots$  it corresponds a sequence of polynomials of Appell type sharing with the standard Bernoulli polynomials many algebraic and combinatorial properties [17]. In particular, the associated numbers satisfy universal Clausen–von Staudt and Kummer congruences, as shown in the Appendix. Also, in the same way as the Riemann zeta function is associated with the Bernoulli polynomials, it is possible to relate the polynomials (49) with a large class of absolutely convergent L-series. Their values at negative integers correspond to the generalized Bernoulli numbers, and they possess as well several interesting number-theoretical properties [16]. Observe that the polynomials (33) belong to the class (49), i.e. the coefficient  $c_i$  are all rational, if the functions  $g(t)$  verify the further condition  $g^{(n)}(0) \in \mathbb{Q}$  for any  $n$ . This is the case, for instance, for the proposed examples of the polynomials (34) and (36). As an immediate consequence, the numbers generated by

$$(51) \quad \frac{t}{(e^t - 1)g(t)} = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!},$$

with the prescribed conditions on  $g(t)$  do satisfy by construction the Clausen–von Staudt and Kummer congruences and many others. These considerations enable us to associate one-variable formal groups with suitable classes of hyperfunctions, as a consequence of the previous construction.

## 6. APPENDIX: CONGRUENCES

The Clausen–von Staudt congruence [8], one of the most beautiful of mathematics, states that

$$(52) \quad B_n + \sum_{p|n} \frac{1}{p} \in \mathbb{Z},$$

where  $B_n$  denotes the  $n$ -th Bernoulli number. This proves the strict link between Bernoulli numbers and prime numbers. Many generalizations of this result have been obtained in the literature in the last decades. In an

attempt to clarify the deep connection between these congruences and algebraic topology, in [5] Clarke proposed the notion of universal Bernoulli numbers  $\widehat{B}_n$ , defined as (50), and proved the remarkable universal von Staudt congruence.

If  $n$  is even, we have

$$(53) \quad \widehat{B}_n \equiv - \sum_{\substack{p-1|n \\ p \text{ prime}}} \frac{c_{p-1}^{n/(p-1)}}{p} \quad \text{mod } \mathbb{Z}[c_1, c_2, \dots];$$

If  $n$  is odd and greater than 1, we have

$$(54) \quad \widehat{B}_n \equiv \frac{c_1^n + c_1^{n-3}c_3}{2} \quad \text{mod } \mathbb{Z}[c_1, c_2, \dots].$$

Like the classical ones, the universal Bernoulli numbers as well play an important role in several branches of mathematics, in particular in complex cobordism theory (see e.g. [2], and [14]), where the coefficients  $c_n$  are identified with the cobordism classes of  $\mathbb{C}P^n$ .

Kummer congruences are also relevant in algebraic topology, and in defining  $p$ -adic extensions of zeta functions. We also have an universal Kummer congruence [1]. Suppose that  $n \neq 0, 1 \pmod{p-1}$ . Then

$$(55) \quad \frac{\widehat{B}_{n+p-1}}{n+p-1} \equiv \frac{\widehat{B}_n}{n} c_{p-1} \quad \text{mod } p\mathbb{Z}_p[c_1, c_2, \dots].$$

Other related congruences can be found in [1].

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